

# 5. CHAIN CONDITIONS

## §5.1. Descending Chain Condition

A ring  $R$  has the **descending chain condition (DCC)** on right ideals if every descending chain of right ideals has a least. DCC on left ideals and DCC on 2-sided ideals are defined similarly.



**Theorem 1:** The class of rings with DCC on right ideals is closed under  $Q$  and  $P$  (and hence under  $\Sigma$ ).

**Proof:**  $Q$ -closure is obvious.

Suppose  $L_1 \geq L_2 \geq \dots$  is a descending chain of right ideals of  $R$ .

Suppose  $I$  and  $R/I$  have DCC on right ideals.

Since  $R/I$  has DCC on right ideals, the descending chain  $(L_1 + I)/I \geq (L_2 + I)/I \geq \dots$  has a least and so there exists  $N_1$  such that for all  $n \geq N_1$ ,  $L_n + I = L_{n+1} + I$  and so

$$L_n \leq L_{n+1} + I.$$

Since  $I$  has DCC on right ideals there exists  $N_2$  such that for all  $n \geq N_2$ ,  $L_n \cap I = L_{n+1} + I$  and so  $L_n \cap I \leq L_{n+1}$ .

Let  $M$  be the maximum of  $N_1$  and  $N_2$  and let  $n \geq M$ .  
Suppose  $x \in L_n$ .

Then  $x = y + z$  where  $y \in L_{n+1}$  and  $z \in I$ .

Thus  $z \in L_n \cap I \leq L_{n+1}$  so  $x \in L_{n+1}$ .

Hence for all  $n \geq M$ ,  $L_n = L_{n+1}$ . 🙌😊

**Example 1:** Let  $R = \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \dots$  be the restricted direct sum of a countably infinite number of copies of  $\mathbb{Z}_2$ . This means that  $R = \{(x_1, x_2, \dots) \mid \text{each } x_i \in \mathbb{Z}_2 \text{ and all but a finite number of them are zero}\}$ . For each  $n$  let  $R_n$  consist of all sequences in  $R$  such that the first  $n$  components are zero. Then  $R_0 > R_1 > \dots$  is a properly descending chain of ideals.

For each  $n$  let  $S_n$  consist of all sequences in  $R$  such that all the components beyond the  $n$ 'th are zero.

Then  $S_1 < S_2 < \dots$  is a properly ascending chain of ideals. So  $R$  has neither ACC nor DCC on ideals. (Since  $R$  is commutative it doesn't matter whether we are considering left ideals, right ideals or two sided ideals – they're all the same.)

Now each  $S_n$  is finite and so has DCC on ideals. So  $R$  is the union of an ascending chain of rings with DCC on ideals, yet it doesn't itself have DCC on ideals. It follows that the class of rings with DCC on any sort of ideals is not  $\cup$ -closed. The same holds for ACC.

**Theorem 2:** For rings with DCC on right ideals every nil right ideal is nilpotent.

**Proof:** Suppose  $L$  is a nil, non-nilpotent right ideal of  $R$  and let  $M \leq L$  be a minimal non-nilpotent right ideal of  $R$ . Hence  $M^2 = M$ .

Let  $N \leq M$  be a minimal right ideal of  $R$  such that:

$$NM \neq 0.$$

Let  $e \in N$  such that  $eM \neq 0$ .

Hence  $eM = N$ .

Let  $f \in M$  such that  $ef = e$ .

Then  $ef^n = e$  for all  $n$ , contradicting the fact that  $L$  is nil.



**Corollary:**  $R_{\mathcal{O}}$  is nilpotent.

**Theorem 3:** If  $R$  has DCC on right ideals  $R_{\mathcal{O}}$  contains all nil left ideals and all nil right ideals.

**Proof:**

Let  $L$  be a nil (and hence nilpotent) right ideal of  $R$ .

Thus  $L + RL$  is a 2-sided nilpotent ideal and so is in  $R_{\mathcal{O}}$ .

Now let  $L$  be a nil left ideal of  $R$  and let  $x \in L, r \in R$ .

Since  $rx$  is nilpotent so is  $xr$ .

Thus  $xR$  is a nil right ideal so  $xR \leq R_{\mathcal{O}}$ .

Hence  $LR \leq R_N$  and so by  $\Sigma$ -closure  $L + LR$  is a nil 2-sided ideal.

Hence  $L \leq L + LR \leq R_{\mathcal{O}}$ .

**Theorem 4:** Suppose  $R$  has DCC on right ideals and  $A$  is a non-nilpotent right ideal of  $R$ . Then  $A$  contains an idempotent.

**Proof:** Let  $M$  be a minimal non-nilpotent right ideal of  $R$  contained within  $A$ .

Since  $M^2 \leq M$  and is a right ideal of  $R$ , and is non-nilpotent,  $M^2 = M$ .

Let  $N$  be a minimal right ideal of  $R$  such that  $NM \neq 0$ .

Then for some  $e \in N$ ,  $eM \neq 0$ . Clearly  $e \neq 0$ .

Now  $eM$  is a right ideal of  $R$  such that  $(eM)M \neq 0$ .

By the minimality of  $M$ ,  $eM = N$ .

Hence there exists  $f \in M$  such that  $ef = e$ .

Clearly  $f$  is non-nilpotent.

Let  $x = f^2 - f$ . Then  $ex = e(f^2 - f)e = 0$ .

Let  $K = \{r \in M \mid er = 0\}$ .

Then  $K$  is a right ideal of  $R$  and  $K < M$ .

By the minimality of  $M$ ,  $K$  is nilpotent and so  $x$  is nilpotent, say  $x^N = 0$ .

Let  $S = \{0 \neq r \in M \mid r^n = 0 \text{ and } r = z^2 - z \text{ with } z \in M\}$ .

Then  $S \subseteq M$ .

Then  $x \in S$ , and so  $S \neq \emptyset$ .

For all  $r \in S$ ,  $r^n = 0$ .

Choose  $s \in S$  such that  $n$  is least.

Clearly  $2 \leq n \leq N$ .

Let  $h = f + s - 2sf$ . Then  $f, s, h \in M$ .

If  $h$  is nilpotent then  $f = h - s + 2sf$  is nilpotent, a contradiction.

(Remember that  $f, h$  and  $s$  commute.)

Hence  $h$  is non-nilpotent, and so is non-zero.

Let  $y = h^2 - h \in M$ .

$$\begin{aligned} \therefore y &= h^2 - h \\ &= f^2 + s^2 + 4f^2s^2 + 2fs - 4f^2s - 4fs^2 - f - s + 2fs \\ &= f^2 + s^2 + 4s^3 + 4fs - 4f^2s - f - s \end{aligned}$$

$$\begin{aligned}
&= s^2 + 4s^3 - 4s^2 \\
&= 4s^3 - 3s^2.
\end{aligned}$$

Hence  $y^k = 0$  for some  $k < n$ .

By the minimality of  $n$ ,  $y = 0$  and so  $h^2 = h$ . 🙌😊

**Corollary:** If  $R$  has DCC on right ideals then the Jacobson radical is nil.

**Proof:** Let  $J$  be the Jacobson radical of  $R$  and suppose that  $x \in R$  is non-nilpotent.

Then,  $xR \leq J$  and so, by Theorem 4,  $xR$  contains an idempotent,  $e$ .

But, then  $-e \in J$  and so is  $RQR$ .

This contradicts Theorem 7 of Chapter 3.

For rings with DCC the diagram in Chapter 4 collapses. The solid line represent the radical class. These classes are all distinct.



## §5.2. Ascending Chain Condition

A ring  $R$  has the **ascending chain condition (ACC)** on right ideals if every ascending chain of right ideals has a greatest.



### Examples 1:

- (1)  $\mathbb{Z}$  has ACC but not DCC.
- (2) The zero ring  $P_2$  on the Prüfer 2-group (Example 5 of Chapter 1) has DCC but not ACC.
- (3)  $M_n(F)$ , the ring of  $n \times n$  matrices over  $F$ , has both ACC and DCC on left or right ideals.
- (4)  $\mathbb{Z} \oplus P_2$  has neither.

**Theorem 5:** If  $R$  has ACC on right ideals every non-zero nil left ideal  $I$  contains a non-zero nilpotent left ideal.

**Proof:** Choose a left ideal  $T$  with  $0 \neq T \leq I$  such that

$$A(T) = \{r \mid tr = 0 \text{ for all } t \in T\}$$

is maximal among such annihilators.

Choose  $0 \neq a \in T$ .

Choose the least  $n$  such that  $Ta^n = 0$ .

Then  $A(T) \leq A(Ta^{n-1})$  so  $A(T) = A(Ta^{n-1})$ .

Hence  $a \in A(T)$  and so  $Ta = 0$ . Thus  $T^2 = 0$ . 🙌😊

**Theorem 6:** If  $R$  is nil-semisimple and has ACC on right ideals there are no non-zero nil left or right ideals of  $R$ .

**Proof:** Suppose  $R$  has a non-zero nil left ideal.

Then, by Theorem 4,  $R$  has a non-zero nilpotent left ideal.

Hence  $R$  has a non-zero nilpotent 2-sided ideal, a contradiction.

Suppose  $R$  has a non-zero nil right ideal.

Then  $R$  has a non-zero nil left ideal, a contradiction. 🙌😊

**Theorem 7:** If  $R$  has ACC on right ideals then every nil left or right ideal is nilpotent.

**Proof:** Suppose  $L$  is a nil left (right) ideal.

Since  $R/R_{\mathcal{N}}$  is nil-semisimple  $L + R_{\mathcal{N}} = R_{\mathcal{N}}$  so  $L \leq R_{\mathcal{N}}$ .

🙌😊

**Corollary:** The classes of nil rings with ACC and nilpotent rings with ACC coincide.

For rings with ACC on right ideals the diagram from Chapter 4 collapses.



